

Quantified preference logic*

Daniel Osherson
Princeton University

Scott Weinstein
University of Pennsylvania

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Abstract

The logic of reason-based preference advanced in Osherson and Weinstein (2012) is extended to quantifiers. Basic properties of the new system are discussed.

1 Introduction

In Osherson and Weinstein (2012) we proposed a logic with binary modal connective \succeq_X that can be read informally as follows.

Let formulas φ and ψ be given along with a collection X of utility scales that measure value along various dimensions. Then $\varphi \succeq_X \psi$ iff the situation that comes to mind when envisioning φ being true is weakly preferred to the situation that comes to mind when envisioning ψ being true, according to the criteria indexed by X .

The utility scales can be conceived as “reasons-for-preference.” For example, p might be dinner tonight with Mitt Romney, and q dinner tonight with Barack Obama. Then if X consists of the two reasons “enjoy lively conversation,” and “influence policy,” it might be true of you that $q \succeq_X (p \wedge q)$. Note that the two reasons cut in different directions: dinner with both would be more lively but offer less chance to influence policy. Affirming the formula thus results from aggregating the two considerations. Of course, you might instead have to evaluate just $p \succeq_X \top$, that is, whether dinner with Mitt is weakly better than your present situation (represented by the tautology \top).

Other examples are discussed in Osherson and Weinstein (2012), and a formal semantics for such sentences is presented and analyzed. The system can be seen as a propositional calculus extended with modal binary connectives. Our present purpose is to show how the propositional part can be replaced with predicate calculus. It is left to Osherson and Weinstein

*Contact: osherson@princeton.edu, weinstein@cis.upenn.edu

(2012) to indicate the previous work that inspired ours, notably, Liu (2008); Dietrich and List (2009). To get started, we specify the languages under consideration, then turn to semantics.

2 Language

2.1 Signatures

A quantified language is built from its “signature.”

- (1) DEFINITION: By a *signature* is meant a pair (\mathbb{L}, \mathbb{U}) where
- (a) \mathbb{L} is a collection of predicates and function symbols of various arities.
 - (b) \mathbb{U} is a nonempty collection of nonempty subsets of natural numbers $(0, 1 \dots)$.

The numbers appearing in $X \in \mathbb{U}$ represent specific reasons for preference such as the desire for lively conversation in our example. A set X of reasons influences preference through aggregation of its members. If $\bigcup \mathbb{U} \in \mathbb{U}$ then preference according to $\bigcup \mathbb{U}$ amounts to preference *tout court*.

2.2 Formulas

We specify the language $\mathcal{L}(\mathbb{L}, \mathbb{U})$ parameterized by the signature (\mathbb{L}, \mathbb{U}) . Formulas are built from the following symbols.

- (a) the members of \mathbb{L} along with the identity sign $=$
- (b) for each $X \in \mathbb{U}$, the binary connective \succeq_X
- (c) the binary connective \wedge and the unary connective \neg
- (d) the quantifier \exists
- (e) the two parentheses, $(,)$
- (f) a denumerable collection $v_0, v_1 \dots$ of individual variables (denoted below by x, y, z).

The set of *terms* is constructed from functions and variables as usual. The set $\mathcal{L}(\mathbb{L}, \mathbb{U})$ of *formulas* is likewise built in the usual way except that we add the clause:

Given $\varphi, \psi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ and $X \in \mathbb{U}$, $\varphi \succeq_X \psi$ also belongs to $\mathcal{L}(\mathbb{L}, \mathbb{U})$.

Moreover, we rely on the following abbreviations.

$\forall x\varphi$	for	$\neg\exists x\neg\varphi$
$(\varphi \vee \psi)$	for	$\neg(\neg\varphi \wedge \neg\psi)$
$(\varphi \rightarrow \psi)$	for	$(\neg\varphi \vee \psi)$
$(\varphi \leftrightarrow \psi)$	for	$((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$
$(\varphi \succeq_{1\dots k} \psi)$	for	$(\varphi \succeq_{\{1\dots k\}} \psi)$
$(\varphi \succ_X \psi)$	for	$(\varphi \succeq_X \psi) \wedge \neg(\psi \succeq_X \varphi)$
$(\varphi \approx_X \psi)$	for	$(\varphi \succeq_X \psi) \wedge (\psi \succeq_X \varphi)$
$(\varphi \preceq_X \psi)$	for	$(\psi \succeq_X \varphi)$
$(\varphi \prec_X \psi)$	for	$(\psi \succ_X \varphi)$
\top	for	$\forall x(x = x)$
\perp	for	$\neg\top$

2.3 Examples

The following formulas serve as illustration.

- (2) (a) $\exists x(Px \succ_X \forall yPy)$
- (b) $\exists xPx \succ_X \forall yPy$

In the domain of people, (2)a affirms that there is someone for whom satisfying P is preferable to everyone satisfying it. This might well be true. For example, from my perspective, it's better that I discover a metric ton of gold than that everyone does (where the reasons encoded in X are basely materialistic). In contrast, (2)b entails that someone getting the gold is better than everyone getting it, which might be false if it doesn't strike me as plausible that I'm the lucky person.

The next example exhibits modal embedding. Consider the domain of potential automobile purchases. Let P refer to the purchase of an efficient gasoline car, and Q to the purchase of a fully electric vehicle. Finally, let the utility indices 1, 2 measure, respectively, ecological value and financial interest. Then

$$\forall x(Px \prec_1 \exists y(Qy \succ_2 Px))$$

says that buying an efficient gasoline car is not as ecologically useful as there being an electric car whose purchase is financially more attractive than buying the gas vehicle. The formula is true if electricity is ecologically superior to gas but consumer choice is based on narrow economic interest. We now turn from intuitive meaning to formal semantics.

3 Semantics

Recall that a signature (\mathbb{L}, \mathbb{U}) consists of vocabulary (\mathbb{L}) and sets of utility indices (\mathbb{U}) .

3.1 Models

(3) DEFINITION: Let a signature (\mathbb{L}, \mathbb{U}) be given. By a *model* for the signature is meant a quintuple $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ where:

- (a) D is a nonempty set, the *domain* of \mathcal{M} .
- (b) \mathbb{W} is a nonempty set of points, the *worlds* of \mathcal{M} .
- (c) t maps $\mathbb{W} \times \mathbb{L}$ to the appropriate set-theoretic objects over D . (For example, if $Q \in \mathbb{L}$ is a binary relation symbol then $t(w, Q)$ is a subset of $D \times D$.) Identity is assigned to $=$.
- (d) u is a function from $\mathbb{U} \times \mathbb{W}$ to the real numbers. For $X, \{i\} \in \mathbb{U}$ we write $u_X(w)$ in place of $u(X, w)$ and $u_i(w)$ in place of $u(\{i\}, w)$.
- (e) s is a function from $\mathbb{W} \times \{A \subseteq \mathbb{W} \mid \emptyset \neq A\}$ such that for all $w \in \mathbb{W}$ and $\emptyset \neq A \subseteq \mathbb{W}$, $s(w, A) \in A$.

Thus, \mathbb{W} corresponds to a set of potential situations; via t , each gives extensions in D to the vocabulary in \mathbb{L} . The function u_X measures the utility of worlds according to the considerations encoded in $X \in \mathbb{U}$. Finally, given a world w_0 and a set A of worlds, s selects a “cognitively salient” member of A , where salience may depend on the vantage point w_0 .

3.2 Propositions

Subsets of worlds are called *propositions*. In the context of a given model, our semantic definition assigns a proposition (subset of \mathbb{W}) to each formula.

Fix a signature (\mathbb{L}, \mathbb{U}) , and let a model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ be given. By an *assignment* (for \mathcal{M}) is meant a map of the individual variables of $\mathcal{L}(\mathbb{L}, \mathbb{U})$ into D . Given a variable x and assignment d , an x *variant* of d is any assignment that differs from d at most in the member of D assigned to x . Assignments are extended to terms of $\mathcal{L}(\mathbb{L}, \mathbb{U})$ in the usual way.

(4) DEFINITION: Let a model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ and assignment d be given. For $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$, the proposition $\varphi[\mathcal{M}, d]$ is defined as follows.

- (a) If φ is $Pt_1 \dots t_n$ for $P \in \mathbb{L}$ and terms $t_1 \dots t_n$ then:
$$\varphi[\mathcal{M}, d] = \{w \in \mathbb{W} \mid \langle d(t_1) \dots d(t_n) \rangle \in t(w, P)\}.$$
- (b) If φ is the negation $\neg\theta$ then $\varphi[\mathcal{M}, d] = \mathbb{W} \setminus \theta[\mathcal{M}, d]$.

- (c) If φ is the conjunction $(\theta \wedge \psi)$ then $\varphi[\mathcal{M}, d] = \theta[\mathcal{M}, d] \cap \psi[\mathcal{M}, d]$.
- (d) If φ is the existential $\exists x \psi$ then $\varphi[\mathcal{M}, d]$ is the set of $w \in \mathbb{W}$ such that $w \in \psi[\mathcal{M}, d']$ for some x variant d' of d .
- (e) If φ has the form $(\theta \succeq_X \psi)$ for $X \in \mathbb{U}$, then $\varphi[\mathcal{M}, d] = \emptyset$ if either $\theta[\mathcal{M}, d] = \emptyset$ or $\psi[\mathcal{M}, d] = \emptyset$. Otherwise:
$$\varphi[\mathcal{M}, d] = \{w \in \mathbb{W} \mid u_X(s(w, \theta[\mathcal{M}, d])) \geq u_X(s(w, \psi[\mathcal{M}, d]))\}.$$

Thus, relative to \mathcal{M} and d , the formula $(\theta \succeq_X \psi)$ expresses the null proposition if evaluating it requires that s choose a world from \emptyset . (Preference makes a covert existential claim in the present theory, namely, that there is something to choose between.) Otherwise $w \in \mathbb{W}$ belongs to the proposition expressed by $(\theta \succeq_X \psi)$ just in case the world chosen by s to represent $\theta[\mathcal{M}, d]$ has greater X -utility than the world chosen by s to represent $\psi[\mathcal{M}, d]$ — where s 's choices depend on the current situation w . Informally, we think of s as choosing the most similar world to w among those available in the proposition at issue.

We extract the assignment-invariant core of a formula's proposition in the standard way.

- (5) DEFINITION: Let $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ and model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ be given. We write $\varphi[\mathcal{M}]$ for the intersection of $\varphi[\mathcal{M}, d]$ over all assignments d .

It follows that for closed $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ (no free variables), $\varphi[\mathcal{M}] = \varphi[\mathcal{M}, d]$ for any assignment d .

3.3 Global modality

We can express global modality in the sense of Blackburn et al. (2001, §2.1) in the following way. Choose any $X \in \mathbb{U}$, and for $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ define:

$$(6) \quad \Box\varphi \stackrel{\text{def}}{=} \neg(\neg\varphi \succeq_X \neg\varphi) \quad \text{and} \quad \Diamond\varphi \stackrel{\text{def}}{=} (\varphi \succeq_X \varphi).$$

Then unwinding clause (4)e of our semantic definition yields:

- (7) PROPOSITION: For all $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$, models $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$, and assignments d :
 - (a) $\Box\varphi[\mathcal{M}, d] \neq \emptyset$ iff $\Box\varphi[\mathcal{M}, d] = \mathbb{W}$ iff $\varphi[\mathcal{M}, d] = \mathbb{W}$.
 - (b) $\Diamond\varphi[\mathcal{M}, d] \neq \emptyset$ iff $\Diamond\varphi[\mathcal{M}, d] = \mathbb{W}$ iff $\varphi[\mathcal{M}, d] \neq \emptyset$.

As an immediate and more perspicuous corollary, we have:

- (8) COROLLARY: For all closed $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$, models $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$:
 - (a) $\Box\varphi[\mathcal{M}] \neq \emptyset$ iff $\Box\varphi[\mathcal{M}] = \mathbb{W}$ iff $\varphi[\mathcal{M}] = \mathbb{W}$.
 - (b) $\Diamond\varphi[\mathcal{M}] \neq \emptyset$ iff $\Diamond\varphi[\mathcal{M}] = \mathbb{W}$ iff $\varphi[\mathcal{M}] \neq \emptyset$.

4 Basic properties of quantified preference logic

4.1 Expressive power of modal formulas

It is worth verifying that our modal vocabulary allows additional propositions to be expressed.

(9) DEFINITION:

- (a) The *modal depth* of formulas is defined inductively. First-order (non-modal) formulas have modal depth zero. If $\varphi, \psi \in \mathbb{L}$ have respective modal depths m, n then $\varphi \succeq_X \psi$ has modal depth $1 + \max\{m, n\}$.
- (b) We say that a model \mathcal{M} *has a modal hierarchy* just in case there are closed formulas $\varphi_0, \varphi_1 \dots$ such that for all $n \geq 0$:
 - (i) φ_n has modal depth n ;
 - (ii) for all closed $\psi \in \mathcal{L}$ of modal depth n or less, $\varphi_{n+1}[\mathcal{M}] \neq \psi[\mathcal{M}]$.

(10) DEFINITION: Let $\mathcal{N} = \langle D, \mathbb{W}, t \rangle$ be the first three components of a model, missing just the utility and selection functions, u, s . Notice that $\langle D, \mathbb{W}, t \rangle$ assigns a proposition $\psi[\mathcal{N}] \subseteq \mathbb{W}$ to each non-modal $\psi \in \mathcal{L}$. We call \mathcal{N} a *normal core* just in case D is countable, \mathbb{W} is countably infinite, and there is non-modal, closed $\psi \in \mathcal{L}$ with $\emptyset \neq \psi[\mathcal{N}] \neq \mathbb{W}$.

Now fix a countable signature (\mathbb{L}, \mathbb{U}) . The following proposition reveals the near ubiquity of modal hierarchies.

(11) PROPOSITION: Let $\mathcal{N} = \langle D, \mathbb{W}, t \rangle$ be a normal core. Then there is a utility function $u : \mathbb{U} \times \mathbb{W} \rightarrow \mathfrak{R}$ and a selection function $s : \mathbb{W} \times \{A \subseteq \mathbb{W} \mid A \neq \emptyset\} \rightarrow \mathbb{W}$ such that the model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ has a modal hierarchy.

PROOF: Choose utility index $X \in \mathbb{U}$, let $\mathcal{N} = \langle D, \mathbb{W}, t \rangle$ be a normal core, and fix closed, non-modal $\psi \in \mathcal{L}$ with $\emptyset \neq \psi[\mathcal{N}] \neq \mathbb{W}$. By replacing ψ with its negation if necessary, we can ensure that $\psi[\mathcal{N}]$ has at least two elements. Let φ_0 be ψ and let φ_{n+1} be $(\top \prec_X \varphi_n)$. Observe that for all $n \in \mathbb{N}$, φ_n has modal depth n . We will define s and u in such a way that $\varphi_0, \varphi_1 \dots$ is a modal hierarchy for $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$.

Let $\{w_0, w_1, \dots\}$ enumerate \mathbb{W} . Since $\psi[\mathcal{N}] = \varphi_0[\mathcal{N}]$ has at least two elements, we may assume without loss of generality that $\{w_0, w_1\} \subseteq \varphi_0[\mathcal{N}]$. Let u be any utility function that meets the conditions:

(12) $u_X(w_0) = 0$ and for all $i > 0$, $u_X(w_i) = 1$.

It remains to specify the selection function s , and to show that it generates a modal hierarchy. This is achieved by inductively defining a sequence of “partial selection” functions s_n , $n \in N$. At stage n , the partial selector s_n defines a partial model $\mathcal{M}_n = \langle D, \mathbb{W}, t, u, s_n \rangle$ which yields a proposition $\chi[\mathcal{M}_n, d]$ for each assignment d , and each $\chi \in \mathcal{L}$ of modal depth n or below. It will be easy to see that for each such χ and d , $\chi[\mathcal{M}_n, d] = \chi[\mathcal{M}, d]$ where $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ with $\bigcup_n s_n \subset s$. Let \mathfrak{P}_n denote the family of nonempty propositions expressed by formulas of modal depth n or below with arbitrary assignments of members of D to their free variables. It is easy to verify that \mathfrak{P}_n is countable. At stage $n = 0$, we let $s_0 = \emptyset$.

For stage $n + 1$, we will define s_{n+1} so that:

- (a) s_{n+1} is defined for every pair (w, X) where $w \in \mathbb{W}$ and $X \in \mathfrak{P}_n$; hence, for every assignment d and $\chi \in \mathcal{L}$ of modal depth n or below, $\chi[\mathcal{M}_n, d]$ is well defined.
- (b) $\varphi_{n+1}[\mathcal{M}_{n+1}] \notin \mathfrak{P}_n$ hence $\varphi_{n+1}[\mathcal{M}] \notin \mathfrak{P}_n$;

Moreover, at every stage n , it will be the case that $\{w_0, w_1\} \subseteq \varphi_n[\mathcal{M}_n]$. In particular, $\{w_0, w_1\} \subseteq \varphi_0[\mathcal{M}_0] = \psi[\mathcal{N}]$ follows from our choice of ψ .

Now we complete stage $n + 1$. For all $w \in \mathbb{W}$, set $s_{n+1}(w, \mathbb{W}) = w_0$ (hence we always draw w_0 from the proposition expressed by \top). For all $w \in \mathbb{W}$ and all $C \in \mathfrak{P}_n - \{\varphi_n[\mathcal{M}_n], \mathbb{W}\}$, choose $s_{n+1}(w, C)$ to be an arbitrary member of C . For the remainder of s_{n+1} , choose $A \subseteq \mathbb{W} - \{w_0, w_1\}$ such that $A \notin \{B - \{w_0, w_1\} \mid B \in \mathfrak{P}_n\}$. Such an A exists because \mathfrak{P}_n is countable. For all $w \in \mathbb{W}$, we define:

$$s_{n+1}(w, \varphi_n[\mathcal{M}_n]) = \begin{cases} w_1 & \text{if } w \in A \cup \{w_0, w_1\} \\ w_0 & \text{otherwise.} \end{cases}$$

It follows immediately from (12) that $\varphi_{n+1}[\mathcal{M}_{n+1}] = A \cup \{w_0, w_1\} \notin \mathfrak{P}_n$. □

A natural question about Proposition (11) is whether modal hierarchies still appear when models satisfy various *frame properties*. To illustrate, model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ is called “reflexive” just in case for all $w \in \mathbb{W}$ and $A \subseteq \mathbb{W}$, if $w \in A$ then $s(w, A) = w$. Reflexivity embodies the idea that the actual world is closer to home than any other world. Several frame properties are examined in Osherson and Weinstein (2012), and also below. In the case of reflexivity, the foregoing proof can be adjusted to show that any normal core can be extended to a reflexive model with modal hierarchy. We leave unexplored the larger project of characterizing the frame properties that allow modal hierarchies, or identifying natural properties that do not.

4.2 Undecidability of satisfaction

Suppose that the signature (\mathbb{L}, \mathbb{U}) contains two unary predicates $P, Q \in \mathbb{L}$. Then it follows from the argument in Kripke (1962) that:

(13) PROPOSITION: The satisfiable subset of $\mathcal{L}(\mathbb{L}, \mathbb{U})$ is not decidable.

Kripke's argument hinges on a mapping from first-order sentences with just the binary relation symbol R to modal sentences that replace Rxy with $\Diamond(Px \wedge Qy)$. On the other hand, the validities are axiomatizable:

(14) PROPOSITION: If the signature is effectively enumerable then so is the set of valid formulas in quantified preference logic.

This fact follows from Proposition (19), below.

4.3 Size of models

Suppose that the signature contains a binary predicate G . Then the upward Löwenheim-Skolem property fails to apply to quantified preference logic. Indeed:

(15) PROPOSITION: There is $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ such that:

- (a) Some model $\langle D, \mathbb{W}, t, u, s \rangle$ with D countable satisfies φ .
- (b) No model $\langle D, \mathbb{W}, t, u, s \rangle$ with D uncountable satisfies φ .

PROOF: Basically, φ says that \prec is a lexicographical order on $D \times D$; such an order cannot be embedded in $\langle \mathbb{R}, < \rangle$ if D is uncountable. For typographical simplicity, we choose $X \in \mathbb{U}$, and write \prec in place of \prec_X .

Specifically, we take φ to be the conjunction of the following formulas.

- (16) (a) $\forall x \forall y (x \neq y \rightarrow ((Gxx \prec Gyy) \vee (Gyy \prec Gxx)))$
- (b) $\forall x_1 y_1 x_2 y_2 ((Gx_1 y_1 \prec Gx_2 y_2) \leftrightarrow ((Gx_1 x_1 \prec Gx_2 x_2) \vee ((x_1 = x_2) \wedge (Gy_1 y_1 \prec Gy_2 y_2))))$

Let a model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ and $w_0 \in \mathbb{W}$ be given with $w_0 \in \varphi[\mathcal{M}]$. We define:

$$X = \{u(s(w_0, Gxx[\mathcal{M}, d(a/x)])) \mid a \in D\}.$$

Then (16)a implies that X (a set of reals) has the same cardinality as D . Define:

$$Y = \{u(s(w_0, Gxy[\mathcal{M}, d(a/x, b/y)])) \mid a, b \in D\}.$$

Then (16)b implies that $\langle Y, < \rangle$ is isomorphic to the lexicographic ordering of $X \times X$.

We leave to the reader the verification that φ is satisfiable in a model with countable domain. On the other hand, suppose that the domain is uncountable, whence X is uncountable. Then the existence of an isomorphism between $\langle Y, < \rangle$ and the lexicographic ordering of $X \times X$ contradicts the separability of the real line. \square

4.4 Preorder models

We can recover the upward Löwenheim-Skolem property by introducing a more general way to compare the value of worlds. Recall that a (*total*) *preorder* is transitive, connected, and reflexive over its domain. Given a signature (\mathbb{L}, \mathbb{U}) , we achieve more generality by replacing u in a model $\langle D, \mathbb{W}, t, u, s \rangle$ with a map \succeq from \mathbb{U} to the set of preorders over \mathbb{W} . [We write \succeq_X for $\succeq(X)$, $X \in \mathbb{U}$.] In such a model $\langle D, \mathbb{W}, t, \succeq, s \rangle$, we evaluate $(\theta \succeq_X \psi)$ according to the following rule, in place of (4)e.

(4)e' If φ has the form $(\theta \succeq_X \psi)$ for $X \in \mathbb{U}$, then $\varphi[\mathcal{M}, d] = \emptyset$ if either $\theta[\mathcal{M}, d] = \emptyset$ or $\psi[\mathcal{M}, d] = \emptyset$. Otherwise:

$$\varphi[\mathcal{M}, d] = \{w \in \mathbb{W} \mid s(w, \theta[\mathcal{M}, d]) \succeq_X s(w, \psi[\mathcal{M}, d])\}.$$

In what follows, we'll call the semantics based on (4)e' *preorder logic*. The original semantics, based on (4)e, will be called *utility logic*. It is easy to see that utility logic is a special case of preorder logic (since assigning utilities to worlds preorders them). Also, it is straightforward to show that the formula φ in the proof of Proposition (15) is satisfied in a preorder model with uncountable domain D . Indeed, the following Löwenheim-Skolem Theorem holds for preorder models.

(17) PROPOSITION: Let $\langle D, \mathbb{W}, t, \succeq, s \rangle$ be a preorder model for a countable signature.

(a) If \mathbb{W} is infinite, then for every infinite cardinal κ there is a preorder model $\mathcal{M}' = \langle D', \mathbb{W}', t', \succeq', s' \rangle$ such that $\text{card}(\mathbb{W}') = \kappa$ and for every sentence φ ,

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M}' \models \varphi.$$

(b) If D is infinite, then for every infinite cardinal κ there is a preorder model $\mathcal{M}' = \langle D', \mathbb{W}', t', \succeq', s' \rangle$ such that $\text{card}(D') = \kappa$ and for every sentence φ ,

$$\mathcal{M} \models \varphi \text{ if and only if } \mathcal{M}' \models \varphi.$$

Despite the greater generality of preorder logic, and the contrast between Propositions (17) and (15), the distinction between utility and preorder models is not discernible by formulas. Indeed:

(18) PROPOSITION: A formula θ is valid in the class of utility models if and only if it is valid in the class of preorder models.

Finally, the next proposition shows that the set of formulas which are valid in preorder models (and hence utility models, by the preceding proposition) is axiomatizable. We assume that the signature is effectively enumerable.

- (19) PROPOSITION: The set of formulas which are valid in preorder models is effectively enumerable.

Proofs of Propositions (17), (18), and (19) are given in the Appendix.

4.5 Generalized preference logic

Reliance on utilities to express preference has a long history in economics e.g., von Neumann and Morgenstern (1944). Proposition (18) provides some justification for this approach inasmuch as the more general preorder logic does not change the class of validities, compared to utility logic. Indeed, validity is preserved in utility logic even if the range of u is limited to the rationals (since every countable preorder is isomorphic to the natural ordering of a subset of rationals).

A yet more general logic is presented in Osherson and Weinstein (2012). We call a quadruple $\langle D, \mathbb{W}, t, r \rangle$ a *generalized model* for signature (\mathbb{L}, \mathbb{U}) if D , \mathbb{W} , and t are as before, and r is a mapping from $\mathbb{W} \times \mathbb{U}$ to the set of preorders over nonempty propositions (nonempty subsets of \mathbb{W}). The selection function s no longer appears since propositions are now compared directly rather than via representative worlds chosen by s . In this kind of model we evaluate $(\theta \succeq_X \psi)$ according to the following variant of (4)e.

- (4)e'' If φ has the form $(\theta \succeq_X \psi)$ for $X \in \mathbb{U}$, then $\varphi[\mathcal{M}, d] = \emptyset$ if either $\theta[\mathcal{M}, d] = \emptyset$ or $\psi[\mathcal{M}, d] = \emptyset$. Otherwise:

$$\varphi[\mathcal{M}, d] = \{w \in \mathbb{W} \mid \theta[\mathcal{M}, d] \text{ comes no earlier than } \psi[\mathcal{M}, d] \text{ in } r(w, X)\}.$$

In Osherson and Weinstein (2012) we analyze the relation between generalized models and utility models in the sentential context. For quantified preference logic, matters are less clear and we leave the following question open.

- (20) OPEN QUESTION: Are the set of formulas valid in utility models the same as the formulas valid in generalized models?

An affirmative answer would provide further evidence for the sufficiency of the utility approach to preference. Of course, validity might be preserved between logics when all their models are considered but break down if attention is limited to certain subsets. The next section illustrates subsets of models that are defined by natural properties. For the remainder of the discussion, only utility models (introduced in Section 3) are at issue.

5 Subclasses of utility models

5.1 Metricity

Many interesting properties of a model $\langle D, \mathbb{W}, t, u, s \rangle$ can be formulated just in terms of \mathbb{W} and s (the model's "frame"). For example, Osherson and Weinstein (2012) consider the following way to express the idea that s chooses "the nearest world."

- (21) DEFINITION: A model $\langle D, \mathbb{W}, t, u, s \rangle$ is *metric* just in case there is a metric $d: \mathbb{W} \times \mathbb{W} \rightarrow \mathfrak{R}$ such that for all $w \in \mathbb{W}$ and $\emptyset \neq A \subseteq \mathbb{W}$, $s(w, A)$ is the unique d -closest member of A to w .

Note that a model is metric only if d -closest worlds exist (there are no chains of worlds ever d -closer to a given world). It is easy to see that in a metric model the set of worlds is countable. There are several properties of models that are implied by metricity. Here we focus on:

- (22) DEFINITION: A model $\langle D, \mathbb{W}, t, u, s \rangle$ is *transitive* just in case for all $A, B, C \subseteq \mathbb{W}$ with $A, B \neq \emptyset$, and $w_0 \in \mathbb{W}$, if $s(w_0, A \cup B) \in A$ and $s(w_0, B \cup C) \in B$ then $s(w_0, A \cup C) = s(w_0, A \cup B)$.

Exploiting our quantificational apparatus, we can write a formula that is true in all transitive models but not valid. We assume that the signature includes the predicate P . For notational ease, we suppress the X on \approx_X .

- (23) PROPOSITION: Let φ be the conjunction of the following formulas.

- (a) $\forall xy(x \neq y \rightarrow (Px \not\approx Py))$
- (b) $\forall xyz((x \neq y \wedge y \neq z \wedge x \neq z) \rightarrow (((Px \vee Py) \approx Px) \wedge ((Py \vee Pz) \approx Py)) \rightarrow (Px \vee Pz) \approx Px)$

Then φ is invalid but valid in the class of transitive models.

The proposition can be viewed as expressing the transitivity of revealed preference, e.g., $(Px \vee Py) \approx Px$ says that Px is chosen from the mutually exclusive options Px, Py .

PROOF: Let model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$, $w_0 \in \mathbb{W}$ and assignment d be given. Let $Px[\mathcal{M}, d] = A$, $Py[\mathcal{M}, d] = B$ and $Pz[\mathcal{M}, d] = C$. If any of $d(x), d(y), d(z)$ are identical or either A or B are empty then we are done. Otherwise, in the presence of (23)a, $(Px \vee Py) \approx Px$ and $(Py \vee Pz) \approx Py$ imply respectively that $s(w_0, A \cup B) \in A$ and $s(w_0, B \cup C) \in B$. So transitivity implies $s(w_0, A \cup C) = s(w_0, A \cup B)$ which entails $w_0 \in (Px \vee Pz) \approx (Px \vee Py)[\mathcal{M}, d]$. So the proposition follows by the transitivity of \approx from $w_0 \in (Px \vee Py) \approx Px[\mathcal{M}, d]$. \square

5.2 Beyond the frame

Rational agents might not be able to discriminate between isomorphic worlds. To formulate this idea, fix a signature (\mathbb{L}, \mathbb{U}) , and let model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ be given. We say that $v, w \in \mathbb{W}$ are *isomorphic* ($v \simeq w$) just in case there is a permutation h of D such that for all $Q \in \mathbb{L}$, h (applied component-wise) maps $t(v, Q)$ onto $t(w, Q)$.

- (24) DEFINITION: Model $\langle D, \mathbb{W}, t, u, s \rangle$ is *utility-invariant* just in case for all isomorphic $v, w \in \mathbb{W}$, $u_X(v) = u_X(w)$ for all $X \in \mathbb{U}$.

This is not a frame property because all components of the model are involved in its formulation. Validity in the utility-invariant models doesn't imply validity in the strict sense. Indeed, we have:

- (25) PROPOSITION: Let signature (\mathbb{L}, \mathbb{U}) be given with \mathbb{L} finite, and distinct $X, Y \in \mathbb{U}$. Then there is invalid $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ that is valid in the class of utility-invariant models.

PROOF: There is $\chi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ such that for all models $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$, $\chi[\mathcal{M}] = \mathbb{W}$ iff $|D| = 2$. Hence, by the finitude of \mathbb{L} and the presence of identity, there is closed, satisfiable $\psi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ such that for all models \mathcal{M} , if $w_1, w_2 \in \psi[\mathcal{M}]$ then $w_1 \simeq w_2$. Let the promised φ be:

$$(\psi \wedge (\psi \succ_X \top) \wedge (\psi \succ_Y \top)) \rightarrow ((\psi \wedge (\psi \succ_X \top)) \approx_X (\psi \wedge (\psi \succ_Y \top))).$$

We indicate why φ is invalid. The antecedent of φ is easily seen to be satisfiable, and a ψ -world satisfying $\psi \wedge (\psi \succ_X \top)$ need not be the same world that satisfies $\psi \wedge (\psi \succ_Y \top)$; and u_X may be chosen to be injective.

On the other hand, suppose that model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ is utility-invariant and let $w_0 \in \mathbb{W}$. Suppose that the antecedent of φ is satisfiable in \mathcal{M} (otherwise, we are done). Then $(\psi \wedge (\psi \succ_X \top))[\mathcal{M}] \neq \emptyset$ and $(\psi \wedge (\psi \succ_Y \top))[\mathcal{M}] \neq \emptyset$. So, let $w_1 = s(w_0, (\psi \wedge (\psi \succ_X \top))[\mathcal{M}])$ and $w_2 = s(w_0, \psi \wedge (\psi \succ_Y \top))[\mathcal{M}]$. Then each of w_1, w_2 satisfies ψ so $w_1 \simeq w_2$. Hence $u_X(w_1) = u_X(w_2)$ by utility-invariance. \square

6 Anonymity

Our final topic concerns the manner in which utilities are associated with formulas. First, a condition is exhibited that makes the utility of a conjunction depend on just the utilities of each conjunct separately. For example, according to this condition the vocabulary appearing in a conjunct is not permitted to influence the utility of the conjunction; rather, the conjunct contributes its utility “anonymously.” A second condition is then introduced that entails a

similar kind of anonymity for the contribution of utility indexes 1 and 2 to the aggregated utility $\{1, 2\}$. The material in this section is inspired by the discussion in Krantz et al. (1971, §7.2).

6.1 Decomposing the utility of conjunctions

Let a signature (\mathbb{L}, \mathbb{U}) be given with predicate $P \in \mathbb{L}$. Conjunctive anonymity with respect to P is expressed by the following formula. (To lighten notation, we suppress $X \in \mathbb{U}$ in subscripts.)

$$(26) \quad \varphi \stackrel{\text{def}}{=} \forall xy((Px \approx Py) \rightarrow \forall z((Px \wedge Pz) \approx (Py \wedge Pz)))$$

The next proposition gives the sense in which φ causes the utility of $Px \wedge Py$ to be a function (F) of the utilities of Px and Py .

(27) PROPOSITION: Let model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ be given with $w_0 \in \varphi[\mathcal{M}]$. Then there is a function $F : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ such that for all assignments d with $Px \wedge Py[\mathcal{M}, d] \neq \emptyset$,

$$u(s(w_0, Px \wedge Py[\mathcal{M}, d])) = F(u(s(w_0, Px[\mathcal{M}, d])), u(s(w_0, Py[\mathcal{M}, d]))).$$

PROOF: For numbers of the form $u(s(w_0, Px[\mathcal{M}, d]))$ and $u(s(w_0, Py[\mathcal{M}, d]))$ define:

$$(28) \quad F(u(s(w_0, Px[\mathcal{M}, d])), u(s(w_0, Py[\mathcal{M}, d]))) \stackrel{\text{def}}{=} u(s(w_0, Px \wedge Py[\mathcal{M}, d])).$$

For all other numbers r_1, r_2 , $F(r_1, r_2)$ is defined arbitrarily. We must show that F is a function. For this purpose, let variable q be given, and suppose that

$$(29) \quad u(s(w_0, Px[\mathcal{M}, d])) = u(s(w_0, Pq[\mathcal{M}, d])).$$

To finish the proof it suffices to show that

$$(30) \quad u(s(w_0, Px \wedge Py[\mathcal{M}, d])) = u(s(w_0, Pq \wedge Py[\mathcal{M}, d])),$$

the second argument of F being treated in the same way. It follows immediately from (29) that $w_0 \in (Px \approx Pq)[\mathcal{M}, d]$, hence by (26)

$$w_0 \in ((Px \wedge Py) \approx (Pq \wedge Py))[\mathcal{M}, d],$$

which implies (30). □

Observe that φ and Proposition (27) can be formulated with disjunction in place of conjunction — or with many other formulas. The proof proceeds in the same way.

6.2 Decomposing a complex utility index

Suppose for this section that the signature (\mathbb{L}, \mathbb{U}) contains unary $P \in \mathbb{L}$ along with $\{1\}, \{2\}, \{1, 2\} \in \mathbb{U}$. Define:

$$(31) \quad \varphi \stackrel{\text{def}}{=} \forall xy (((Px \approx_1 Py) \wedge (Px \approx_2 Py)) \rightarrow (Px \approx_{1,2} Py)).$$

Then φ implies that the contributions of 1 and 2 to the complex utility index $\{1, 2\}$ can be separated then brought back together via a binary mapping on \mathfrak{R} . Specifically:

(32) PROPOSITION: Let model $\mathcal{M} = \langle D, \mathbb{W}, t, u, s \rangle$ be given with $w_0 \in \varphi[\mathcal{M}]$. Then there is a function $F : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ such that for all assignments d :

$$u_{1,2}(s(w_0, Px[\mathcal{M}, d])) = F(u_1(s(w_0, Px[\mathcal{M}, d])), u_2(s(w_0, Px[\mathcal{M}, d]))).$$

PROOF: Call a pair $(p, q) \in \mathfrak{R}^2$ *critical* just in case there is an assignment d such that

$$(33) \quad \begin{aligned} \text{(a)} \quad & p = u_1(s(w_0, Px[\mathcal{M}, d])) \\ \text{(b)} \quad & q = u_2(s(w_0, Px[\mathcal{M}, d])). \end{aligned}$$

Let $F : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ be such that for any critical pair (p, q) as in (33), $F(p, q) = u_{1,2}(s(w_0, Px[\mathcal{M}, d]))$. The behavior of F on noncritical pairs is arbitrary. Suppose that for some assignment d' :

$$(34) \quad \begin{aligned} \text{(a)} \quad & p = u_1(s(w_0, Px[\mathcal{M}, d'])) \\ \text{(b)} \quad & q = u_2(s(w_0, Px[\mathcal{M}, d'])). \end{aligned}$$

To verify that F is a function, thereby completing the proof, we must show that

$$(35) \quad u_{1,2}(s(w_0, Px[\mathcal{M}, d])) = u_{1,2}(s(w_0, Px[\mathcal{M}, d'])).$$

Let y be a variable distinct from x , and let $d'' = d(d'(x)/y)$. From (33) and (34) we infer: $w_0 \in Px \approx_1 Py[\mathcal{M}, d'']$ and $w_0 \in Px \approx_2 Py[\mathcal{M}, d'']$. From (31) we then obtain $w_0 \in Px \approx_{1,2} Py[\mathcal{M}, d'']$ from which (35) is an immediate consequence. \square

Appendix

We present proofs of Propositions (17), (18), and (19) from Section 4.4. All three proofs elaborate a construction that appears in the demonstration of Theorem (55) in Osherson and Weinstein (2012). Specifically, the earlier construction can be adapted to show that there is an effective translation from sentences $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ to formulas $\varphi^\dagger(x)$ of first-order logic, and a map from preorder models $\mathcal{M} = \langle D, \mathbb{W}, t, \succeq, s \rangle$ to relational structures $\mathcal{F}_\mathcal{M}$ such that

$$(36) \quad w \in \varphi[\mathcal{M}] \text{ iff } \mathcal{F}_{\mathcal{M}} \models \varphi^\dagger[w].$$

Moreover, assuming that (\mathbb{L}, \mathbb{U}) is recursive, there is a recursively axiomatizable first-order theory T in the signature of $\mathcal{F}_{\mathcal{M}}$ such that

$$(37) \quad \text{for every preorder model } \mathcal{M}, \mathcal{F}_{\mathcal{M}} \models T$$

and

$$(38) \quad \text{for every first-order structure } A, \text{ if } A \models T, \text{ then for some preorder model } \mathcal{M}, A = \mathcal{F}_{\mathcal{M}}.$$

Proposition (19) now follows from the completeness theorem for first-order logic, since (36), (37), and (38) imply that $\varphi \in \mathcal{L}(\mathbb{L}, \mathbb{U})$ is valid in preorder logic if and only if $\forall x \varphi^\dagger(x)$ is a consequence of T . In like fashion, Proposition (17) follows from the Löwenheim-Skolem Theorem for first-order logic. Proposition (18) now follows immediately, since every countable preorder model is induced by a corresponding utility model, a consequence of the fact that the rational numbers are universal among countable linear orders. \square

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